Topological Multivortices Solutions of the Self-Dual Maxwell-Chern-Simons-Higgs System

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Abstract

We study existence and various behaviors of topological multivortices solutions of the relativistic self-dual Maxwell-Chern-Simons-Higgs system. We first prove existence of general topological solutions by applying variational methods to the newly discovered minimizing functional. Then, by an iteration method we prove existence of topological solutions satisfying some extra conditions, which we call admissible solutions. We establish asymptotic exponential decay estimates for these topological solutions. We also investigate the limiting behaviors of the admissible solutions as parameters in our system goes to some limits. For the Abelian Higgs limit we obtain strong convergence result, while for the Chern-Simons limit we only obtained that our admissible solutions are weakly approximating one of the Chern-Simons solutions.

Introduction

Since the pioneering works by Ginzburg and Landau on the superconductivity there are many studies on the Abelian Higgs system[10], [12](and references therein). In particular in [10] Jaffe and Taubes established the unique existence of general finite energy multi-vortices solutions for the Bogomol'nyi equations. (See also [12] for more constructive existence proof together with explicit numerical solutions.)
More recently, motivated largely by the physics of high temperature superconductivity the self-dual Chern-Simons system(hereafter Chern-Simons system) was modeled in [2] and [3].(See [1] for a general survey.) The general existence theorem of topological solutions for the corresponding Bogomol'nyi equations was established in [11] by a variational method, and in [8] by an iteration argument. For the nontopological boundary condition we have only general existence result for the radial solutions for vortices in a single point[9].
We recall that in the Lagrangian of the Chern-Simons system there is no Maxwell term appearing in the Abelian Higgs system, while the former includes the Chern-Simons term which is not present in the later. Naive inclusion of both of the two terms in the Lagrangian makes the system non self-dual(i.e. there is no Bogomol'nyi type equations for the nontrivial global minimizer of the energy functional.) In [5], however, a self-dual system including both of the Maxwell and the Chern-Simons terms, so called (relativistic) *self-dual Maxwell-Chern-Simons-Higgs system* was successfully modeled using the $N = 2$ supersymmetry argument[4], [6]. It was found that we need extra neutral scalar field to make the system self-dual.
In this paper we first prove general existence theorem for topological multi-vortex solutions of the corresponding Bogomol'nyi equations of this system by a variational method. Then, using an iteration argument we constructively prove existence of a class of solutions enjoying some extra conditions. We call topological solutions satisfying these extra conditions the *admissible topological solutions*. We prove asymptotic exponential decay estimates for the various terms in our Lagrangian for the general topological solutions. One of the most interesting facts for the admissible topological solutions is that these solutions are really "interpolated" between the Abelian Higgs solution and the Chern-Simons solutions in the the following sense: for fixed electric charge, when the Chern-Simons coupling constant goes

to zero, our solution converges to the solution of the Abelian Higgs system. The convergence in this case is very strong. On the other hand, when both the Chern-Simons coupling constant and the electric charge goes to infinity with some constraints between the two constants, we proved that our solution is "weakly approximately" satisfying the Bogomol'nyi equations for Chern-Simons system. In the existence proof for the admissible topological solutions, although we used iteration method in $\mathbf{R}^2$ directly, we could start iteration in a bounded domain with suitable boundary condition to obtain a solution in that domain, and then enlarge this domain to the whole of $\mathbf{R}^2$ as is done in [8], and [12] in a much simpler case than ours. In this way it would be possible to obtain an explicit numerical solution.
The organization of this paper is following.
In the section 1 we introduce the action functional for the self-dual Maxwell-Chern-Simons-Higgs system, and deduce a system of second order elliptic partial differential equations which is a reduced version of the Bogomol'nyi system. In the section 2 by a variational method we prove a general existence of topological solutions. Then, we introduce the notion of admissible topological solution. In the section 3, 4 and 5 we prove existence of admissible topological solutions by an iteration method. In the section 5, in particular, we establish exponential decay estimates for our solutions. In the section 6 we prove strong convergence of the admissible topological solutions to the Abelian Higgs solution. The last section considers the Chern-Simons limit, and we prove that admissible topological solutions are weakly consistent to the Chern-Simons equation in this limit.
(*After finishing this work, we found that there was a study on the non relativistic version of our model by Spruck-Yang in [7].*)

1 Preliminaries

The Lagrangian density for the (relativistic) self-dual Maxwell-Chern-Simons-Higgs system in $(2+1)$-D Lagrangian density modeled by C. Lee *et al*[5] is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\kappa}{4}\epsilon^{\mu\nu\lambda}F_{\mu\nu}A_\lambda + D_\mu\phi D^\mu\phi^* \\ &\quad +\frac{1}{2}\partial_\mu N\partial^\mu N - q^2N^2|\phi|^2 - \frac{1}{2}(q|\phi|^2+\kappa N-q)^2 \qquad (1)\end{aligned}$$

where ϕ is a complex scalar field, N is a real scalar field, $A = (A_0, A_1, A_2)$ is a vector field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu = \partial_\mu - iqA_\mu$, $\mu = 0, 1, 2$, $\partial_0 = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$, $j = 1, 2$, $q > 0$ is the charge of electron, and $\kappa > 0$ is a coupling constant for Chern-Simons term. The action functional for this system is given by

$$\mathcal{A} = \int_{\mathbf{R}^3} \mathcal{L} dx. \tag{2}$$

The static energy functional for the above system is

$$\begin{aligned} \mathcal{E} &= \int_{\mathbf{R}^2} \left\{ |D_0\phi|^2 + |D_j\phi|^2 + \frac{1}{2}F_{j0}^2 + \frac{1}{2}F_{12}^2 + \frac{1}{2}(\partial_j N)^2 \right. \\ &\qquad \left. + q^2 N^2 |\phi|^2 + \frac{1}{2}(q|\phi|^2 + \kappa N - q)^2 \right\} dx. \end{aligned} \tag{3}$$

with the Gauss law constraint

$$(-\Delta + 2q^2|\phi|^2)A_0 = -\kappa F_{12}. \tag{4}$$

This is the Euler-Lagrange equation with variation of the action taken with respect to A_0. Integrating by parts, using (4), we obtain from the energy functional

$$\begin{aligned} \mathcal{E} &= \int_{\mathbf{R}^2} \left\{ |(D_1 \pm iD_2)\phi|^2 + |D_0\phi \mp iq\phi N|^2 + \frac{1}{2}(F_{j0} \pm \partial_j N)^2 \right. \\ &\qquad \left. + \frac{1}{2}|F_{12} \pm (q|\phi|^2 + \kappa N - q)|^2 \right\} dx \pm q \int_{\mathbf{R}^2} F_{12} dx. \end{aligned} \tag{5}$$

This implies the lower bound for the energy

$$\mathcal{E} \geq q \left| \int_{\mathbf{R}^2} F_{12} dx \right|,$$

which is saturated by the solutions of the equations(the Bogomol'nyi equations) for (ϕ, A, N)

$$A_0 = \mp N \tag{6}$$

$$(D_1 \pm iD_2)\phi = 0 \tag{7}$$

$$F_{12} \pm (q|\phi|^2 + \kappa N - q) = 0 \tag{8}$$

$$\Delta A_0 = \pm(\kappa q(1 - |\phi|^2) + \kappa^2 A_0) + 2q^2|\phi|^2 A_0 \tag{9}$$

where the upper(lower) sign corresponds to positive(negative) values of $\int_{\mathbf{R}^2} F_{12}dx$, and (9) follows from the Gauss law combined with (8). If (ϕ, A, N) is a solution that makes $\mathcal{E}$ finite, then either

$$\phi \to 0 \quad \text{and} \qquad N = -A_0 \to \frac{q}{\kappa},$$

or

$$|\phi|^2 \to 1 \quad \text{and} \qquad N = -A_0 \to 0$$

as $|x| \to \infty$. The former is called non-topological, and the latter is called topological. In this paper we are considering only topological boundary condition. We set

$$\phi = e^{\frac{1}{2}(u+i\theta)}, \quad \theta = \sum_{j=1}^{m} 2n_j \arg(z - z_j), \quad n_j \in Z^+,$$

where $z = (x, y)$ is the canonical coordinates in $\mathbf{R}^2$, and each $z_j = (x_j, y_j)$ is a zero of ϕ with winding number n_j, which corresponds to the multiplicity of the $j-$th vortex. After similar reduction procedure similar to [10], we obtain the equations(we have chosen the upper sign.)

$$\Delta u = 2q^2(e^u - 1) - 2q\kappa A_0 + 4\pi \sum_{j=1}^{m} n_j \delta(z - z_j) \tag{10}$$

$$\Delta A_0 = \kappa q(1 - e^u) + (\kappa^2 + 2q^2 e^u)A_0 \tag{11}$$

with the boundary condition (14). We define

$$f = \sum_{j=1}^{m} n_j \ \ln\left(\frac{|z - z_j|^2}{1 + |z - z_j|^2}\right),$$

and we set $u = v + f$ to remove the singular inhomogeneous term in (10). Then (10) and (11) become

$$\Delta v = 2q^2(e^{v+f} - 1) - 2q\kappa A_0 + g, \tag{12}$$

$$\Delta A_0 = \kappa q(1 - e^{v+f}) + (\kappa^2 + 2q^2 e^{v+f})A_0 \tag{13}$$

with the boundary condition

$$\lim_{|z|\to\infty} v = 0, \qquad \lim_{|z|\to\infty} A_0 = 0, \tag{14}$$

where

$$g = \sum_{j=1}^{m} \frac{4n_j}{(1 + |z - z_j|^2)^2}$$

2 Existence of a Variational Solution

Solving (12) for A_0, and substituting this into (13), we obtain

$$\begin{aligned}\Delta^2 v \quad &- \quad (\kappa^2 + 4q^2 e^{v+f})\Delta v + 4q^4 e^{v+f}(e^{v+f} - 1) \\ &= \quad 2q^2|\nabla(v+f)|^2 e^{v+f} - 4q^2 g e^{v+f} + \Delta g - \kappa^2 g \qquad (15)\end{aligned}$$

If we formally set $\kappa = 0$ in this equation, then we have

$$\Delta(\Delta v - 2q^2(e^{v+f} - 1) - g) = 0$$

which, if we ask $v \in H^2(\mathbf{R}^2)$, recovers the Abelian Higgs system studied in [10]. On the other hand, if we take the limit $\kappa, q \to \infty$ with $q^2/\kappa = l$ fixed number, then after formally dropping the lower order terms in $o(q)$ and $o(\kappa)$, we obtain

$$\Delta v = 4l^2 e^{v+f}(e^{v+f} - 1) + g.$$

This is the equation corresponding to the pure Chern-Simons system studied in [8], [11], etc.
Later in this paper we provide rigorous justifications for these two limiting behaviors of the solutions. By a direct calculation we find that the equation (15) is a variational equation of the following functional.

$$\begin{aligned}\mathcal{F}(v) \quad &= \quad \int \left[\frac{1}{2}|\Delta v|^2 - (\Delta g - \kappa^2 g)v + 2q^4(e^{v+f} - 1)^2 \right. \\ &\quad \left. + \frac{1}{2}\kappa^2|\nabla v|^2 + 2q^2 e^{v+f}|\nabla(v+f)|^2 \right] dx \qquad (16)\end{aligned}$$

The above functional is well-defined in $H^2(\mathbf{R}^2)$ since $e^f|\nabla f|^2 \in L^1(\mathbf{R}^2)$. We now prove existence of solution of (15) in $H^2(\mathbf{R}^2)$. Further regularity then $v \in H^2(\mathbf{R}^2)$ follows from the standard regularity results for the nonhomogeneous biharmonic equations.

Theorem 1 *The functional (16) is coercive, and weakly lower semicontinuous in $H^2(\mathbf{R}^2)$, and thus there is a global minimizer of the functional (16) in $H^2(\mathbf{R}^2)$.*

Proof: If a sequence $\{v_k\}$ converges to v weakly in $H^2(\mathbf{R}^2)$, $v_k \to v$ strongly in $L^\infty(B_R)$ and in $H^1(B_R)$ for any ball $B_R = B(0,R) \subset \mathbf{R}^2$ by Rellich's compactness theorem. Thus we observe that to prove the

lower semi-continuity of the functional (16), it is sufficient to prove the lower semi-continuity of $\int_{\mathbf{R}^2} e^{v+f}|\nabla(v+f)|^2dx$. We have

$$\begin{aligned}\liminf_{k\to\infty}\int_{\mathbf{R}^2} e^{v_k+f}|\nabla(v_k+f)|^2dx &\geq \liminf_{k\to\infty}\int_{B_R} e^{v_k+f}|\nabla(v_k+f)|^2dx \\ &= \int_{B_R} e^{v+f}|\nabla(v+f)|^2dx.\end{aligned}$$

Letting $R\to\infty$, we obtain the desired weak lower semi-continuity. On the other hand, we note that the coercivity of $\mathcal{F}$ in $H^2(\mathbf{R}^2)$, is a simple corollary of the inequality:

$$\|v\|^2_{L^2(\mathbf{R}^2)} \leq C(1+\|\Delta v\|^2_{L^2(\mathbf{R}^2)} + \|e^{v+f}-1\|^2_{L^2(\mathbf{R}^2)}), \tag{17}$$

since we have

$$|\int(\Delta g-\kappa^2 g)vdx| \leq \frac{C}{\epsilon}+\epsilon\|v\|^2_{L^2(\mathbf{R}^2)}$$

for any $\epsilon>0$, and by the Calderon-Zygmund inequality we have

$$\|D^2v\|_{L^2(\mathbf{R}^2)} \leq C\|\Delta v\|_{L^2(\mathbf{R}^2)}.$$

For the proof of (17), we just recall the inequality (4.10) in [11], which immediately implies;

$$\|v\|^2_{L^2(\mathbf{R}^2)} \leq C(1+\|\nabla v\|^2_{L^2(\mathbf{R}^2)} + \|e^{v+f}-1\|^2_{L^2(\mathbf{R}^2)}).$$

Now, for any $\eta>0$ we have

$$\int|\nabla v|^2dx = -\int v\Delta v dx \leq \eta\|v\|^2_{L^2(\mathbf{R}^2)} + C_\eta\|\Delta v\|^2_{L^2(\mathbf{R}^2)}.$$

Taking η small enough, (17) follows. This completes the proof of the theorem. $\square$

Proposition 1 *Let (v, A_0) be any topological solution of (12)-(13), and v_a^q be the finite energy solution of the Abelian Higgs system. Then the following conditions are equivalent.*

(i) $A_0 \leq 0$

(ii) $v \leq -f$

(iii) $A_0 \geq \frac{q}{\kappa}(e^{v+f} - 1)$

(iv) $v \leq v_a^q$.

Proof:
(i)⇒(ii), (i)⇒(iv): We assume $A_0 \leq 0$. Let v_a^q be the solution of the Abelian Higgs system, i.e. v_a^q satisfies

$$\Delta v_a^q = 2q^2(e^{v_a^q+f} - 1) + g. \tag{18}$$

The existence and uniqueness of $v_a^q \in H^2(\mathbf{R}^2) \cap C^\infty(\mathbf{R}^2)$ satisfying $v_a^q \leq -f$ is well-known[10]. From (12) with $A_0 \leq 0$ we have

$$\Delta v \geq 2q^2(e^{v+f} - 1) + g.$$

Thus,
$$\Delta(v_a^q - v) \leq 2q^2(e^{v_a^q f} - e^{v+f}) = 2q^2 e^{\lambda+f}(v_a^q - v)$$

by the mean value theorem, where λ is between v and $v^{\kappa,q}$. By the maximum principle we have

$$v \leq v_a^q \leq -f.$$

(ii)⇒(i): We assume $v \leq -f$. From (13) we have

$$\Delta A \geq (\kappa^2 + 2q^2 e^{v+f})A.$$

Thus (i) follows from the maximum principle.
(i)⇒(iii): We assume $A_0 \leq 0$. Set $G = \frac{q}{\kappa}(1 - e^{v+f})$. Then, we compute.

$$\begin{aligned}
\Delta G &= -\frac{q}{\kappa}|\nabla(v+f)|^2 e^{v+f} - \frac{q}{\kappa}\Delta(v+f)e^{v+f} \\
&\leq -\frac{q}{\kappa}e^{v+f}[2q^2(e^{v+f} - 1) - 2q\kappa A_0] \\
&= 2q^2 e^{v+f} G + 2q^2 A_0
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\Delta(G + A_0) &\leq (\kappa^2 + 2q^2 e^{v+f})(G + A_0) + 2q^2 A_0 \\
&\leq (\kappa^2 + 2q^2 e^{v+f})(G + A_0).
\end{aligned}$$

Since $G \to 0$ as $|z| \to \infty$, we have $G + A \geq 0$ by the maximum principle.

(iv)⇒(ii): This is obvious, and included in the above proof.
(iii)⇒(i): Assuming (iii), we have from (13)

$$\Delta A_0 \geq 2q^2 e^{v+f} A_0.$$

Thus, (i) follows again by the maximum principle. This completes the proof of the proposition. $\square$

Definition 1 *We call a topological solution* (v, A_0) *satisfying any one of the four conditions in Proposition 1 by an admissible topological solution.*

3 Iteration Scheme

In this section we construct an approximate multi-vortices solution sequence of our Bogomol'nyi equations by an iteration scheme. Later this approximate solution sequence will be found to converge to an admissible topological solution. Our iteration scheme is similar to that of [8], but is substantially extended in form.

Definition 2 *We set* $v^0 = v_a^q$, $A_0^0 = 0$, *where* v_a^q *is the finite energy solution of the Abelian Higgs system. Define* $(v^i, A_0^i) \in H^2(\mathbf{R}^2) \cap C^\infty(\mathbf{R}^2)$, $i \geq 1$ *iteratively as follows:*
First define v^i *from* (v^{i-1}, A_0^{i-1}) *by solving:*

$$(\Delta - d)v^i = 2q^2(e^{v^{i-1}+f} - 1) - 2q\kappa A_0^{i-1} + g - dv^{i-1}, \tag{19}$$

and then define A_0^i *from* (v^i, A_0^{i-1}) *by solving:*

$$(\Delta - \kappa^2 - 2q^2 e^{v^i+f} - d)A_0^i = \kappa q(1 - e^{v^i+f}) - dA_0^{i-1}. \tag{20}$$

Here, $d \geq 2q^2$ *is a constant that will be fixed later.*

Lemma 1 *The scheme (19)-(20) is well-defined, and the iteration sequence* (v^i, A_0^i) *satisfies the monotonicity, i.e.*

$$\begin{aligned} v^i &\leq v^{i-1} \leq \cdots \leq v^0 \leq -f \\ A_0^i &\leq A_0^{i-1} \leq \cdots \leq A_0^0 = 0. \end{aligned}$$

Proof: We proceed by an induction. For $i = 1$ we have from (19)

$$(\Delta - d)v^1 = 2q^2(e^{v^0+f} - 1) + g - dv^0$$

On the other hand v^0 satisfies

$$\Delta v^0 = 2q^2(e^{v^0+f} - 1) + g.$$

Thus we have

$$(\Delta - d)(v^1 - v^0) = 0.$$

From this we obtain $v^1 = v^0 \in H^2(\mathbf{R}^2) \cap C^\infty(\mathbf{R}^2)$, and obviously

$$v^1 \leq v^0 \leq -f.$$

Now from (20) for $i = 1$ we have

$$(\Delta - \kappa^2 - 2q^2 e^{v^1+f} - d)A_0^1 = \kappa q(1 - e^{v^1+f}). \tag{21}$$

Using the mean value theorem, we obtain

$$\int_{\mathbf{R}^2}(1 - e^{v^1+f})^2 dx \leq \int_{\mathbf{R}^2}(v^1 + f)^2 e^{\lambda+f} dx \leq \int_{\mathbf{R}^2}(v^1 + f)^2 dx < \infty,$$

where $v^1 < \lambda < -f$, and we used the fact $f \in L^2(\mathbf{R}^2)$. We also have $0 \leq e^{v^1+f} < 1$. Thus, by the standard result of the linear elliptic theory the equation (21) defines $A_0^1 \in H^2(\mathbf{R}^2) \cap C^\infty(\mathbf{R}^2)$.
Furthermore, since $\kappa q(1 - e^{v^1+f}) \geq 0$, by the maximum principle applied to (21) we have

$$A_0^1 \leq A_0^0 = 0.$$

Thus Lemma 1 is true for $i = 1$. Suppose the lemma is true up to $i-1$. Clearly (19)-(20) define $(v^i, A_0^i) \in H^2 \cap C^\infty(\mathbf{R}^2)$ from (v^{i-1}, A_0^{i-1}). We only need to observe that

$$\int_{\mathbf{R}^2}(e^{v^i+f} - 1)^2 dx = \int_{\mathbf{R}^2}(v^i + f)^2 e^{\lambda+f} dx < \infty$$

and

$$0 \leq e^{v^i+f} \leq 1$$

if $v^{i-1} \in L^2(\mathbf{R}^2)$ and $v^{i-1} \leq -f$. We also have

$$\begin{aligned}(\Delta - d)(v^i - v^{i-1}) &= 2q^2(e^{v^{i-1}+f} - e^{v^{i-2}+f}) - 2q\kappa(A^{i-1} - A^{i-2}) \\ &\quad -d(v^{i-1} - v^{i-2}) \\ &\geq d(e^{v^{i-1}+f} - e^{v^{i-2}+f} - (v^{i-1} - v^{i-2})) \\ &= d(e^{f+\lambda} - 1)(v^{i-1} - v^{i-2}),\end{aligned}$$

where $v^{i-1} + f \leq \lambda \leq v^{i-2} + f$, and we used the mean value theorem. Since

$$e^{w^i} \leq 1 \quad \text{and} \quad v^{i-1} - v^{i-2} \leq 0$$

by induction hypothesis, we obtain

$$(\Delta - d)(v^i - v^{i-1}) \geq 0.$$

Applying maximum principle again, we have $v^i \leq v^{i-1}$. On the other hand,

$$\begin{aligned}(\Delta - d)(A_0^i - A_0^{i-1}) &= -\kappa q(e^{v^i+f} - e^{v^{i-1}+f}) \\ &\quad +(\kappa^2 + 2q^2 e^{v^i+f})(A_0^i - A_0^{i-1}) \\ &\quad +2q^2(e^{v^i+f} - e^{v^{i-1}+f})A_0^{i-1} - d(A_0^{i-1} - A_0^{i-2}) \\ &\geq (\kappa^2 + 2q^2 e^{v^i+f})(A_0^i - A_0^{i-1}),\end{aligned}$$

where we used the assumption that our lemma holds up to $i-1$, and $v^i \leq v^{i-1}$. Therefore, by the maximum principle,

$$A_0^i \leq A_0^{i-1}$$

Lemma 1 is thus proved. $\square$

Lemma 2 *The iteration sequence (v^i, A_0^i) satisfies the inequality*

$$A_0^i \geq \frac{q}{\kappa}(e^{v^i+f} - 1).$$

for each $i = 0, 1, 2, \cdots$.

Proof: We use induction again. Lemma 2 is true for $i = 1$. We set $G^i = \frac{q}{\kappa}(1 - e^{v^i+f})$. Suppose Lemma 2 holds for $i - 1$, then

$$\begin{aligned}
\Delta G^i &= -\frac{q}{\kappa}\nabla\cdot(\nabla(v^i + f)e^{v^i+f}) \\
&= -\frac{q}{\kappa}|\nabla(v^i + f)|^2 e^{v^i+f} - \frac{q}{\kappa}\Delta(v^i + f)e^{v^i+f} \\
&\leq -\frac{q}{\kappa}e^{v^i+f}(2q^2(e^{v^{i-1}+f} - 1) - 2q\kappa A_0^{i-1} + d(v^i - v^{i-1})) \\
&\leq 2q^2 e^{v^{i-1}+f}G^i + \frac{q}{\kappa}(e^{v^i+f} - e^{v^{i-1}+f} - \frac{q}{\kappa}de^{v^i+f}(v^i - v^{i-1}) \\
&\leq 2q^2 e^{v^{i-1}+f}G^i - \frac{q}{\kappa}de^{v^i+f}(v^i - v^{i-1})
\end{aligned}$$

by $A_0^i \leq 0$, $i \geq 0$. Therefore

$$\begin{aligned}
(\Delta - d)(G^i + A_0^i) &\leq (\kappa^2 + 2q^2 e^{v^i+f})(A_0^i + G^i) \\
&\quad -dG^i - dA_0^{i-1} - \frac{q}{\kappa}de^{v^i+f}(v^i - v^{i-1}) \\
&\leq (\kappa^2 + 2q^2 e^{v^i+f})(A_0^i + G^i) \\
&\quad -d(A_0^{i-1} + G^i + \frac{q}{\kappa}(e^{v^i+f} - e^{v^{i-1}+f}),
\end{aligned}$$

where we used the mean value theorem in the last step,and used the fact $v^i \leq v^{i-1}$. Rewriting it, we have

$$(\Delta - \kappa^2 - 2q^2 e^{v^i+f} - d)(G^i + A_0^i) \leq -d(A_0^{i-1} + G^{i-1}) \leq 0$$

by the induction hypothesis. By maximum principle we have $A_0^i + G^i \geq 0$. This completes the proof of Lemma 2. □

4 Monotonicity of $\mathcal{F}(v^i)$

In this section we will prove the following:

Lemma 3 *Let $\{v^i\}$ given as in Definition 2 and $\mathcal{F}(v)$ is given in (16). We have*

$$\mathcal{F}(v^i) \leq \mathcal{F}(v^{i-1}) \leq \cdots \leq \mathcal{F}(v^0). \tag{22}$$

To prove this we firstly begin with:

Lemma 4 *Let (v^i, A_0^i) be as in Definition 2, then*

$$\int_{\mathbf{R}^2}\left[|1-e^{v^i+f}+\frac{\kappa}{q}A_0^i|+\frac{d}{2q^2}|v^i-v^{i+1}|\right]dx=\frac{2\pi}{q^2}\sum_{j=1}^m n_j$$

for all $i \geq 0$.

From Lemma 1 and 2 we have

$$1-e^{v^i+f}+\frac{\kappa}{q}A_0^i,\ \ v^i-v^{i+1}\geq 0.$$

We only need to prove

$$\int_{\mathbf{R}^2}\left[1-e^{v^i+f}+\frac{\kappa}{q}A_0^i+\frac{d}{2q^2}(v^i-v^{i+1})\ \right]dx=\frac{1}{2q^2}\int_{\mathbf{R}^2}g\ dx.$$

Fix $R>0$. Integrating (19) over $B_R=\{|z|<R\}$, we obtain

$$\begin{aligned}&\int_{B_R}\left[(1-e^{v^i+f}+\frac{\kappa}{q}A_0^i)+\frac{d}{2q^2}(v^i-v^{i+1})\ \right]dx\\&=\frac{1}{2q^2}\int_{B_R}(g-\Delta v^{i+1})\ dx\end{aligned}\tag{23}$$

By divergence theorem

$$\int_{B_R}\Delta v^i=\int_{\partial B_R}\frac{\partial v^i}{\partial r}d\sigma.$$

We note that $v^i\in H^1(\mathbf{R}^2)$. Thus

$$\int_{\partial B_R}|\nabla v^i|d\sigma\leq\left(2\pi R\int_{\partial B_R}|\nabla v^i|^2d\sigma\right)^{1/2}\tag{24}$$

by Hölder's inequality. Let

$$H(r)=\int_{\partial B_r}|\nabla v^i|^2d\sigma,$$

then

$$\int_{\mathbf{R}^2}|\nabla v^i|^2\ dx=\int_0^\infty H(r)dr<+\infty$$

Therefore there exists an increasing sequence of radii, $\{r_k\}_{k=1}^\infty$, such that

$$\lim_{k\to\infty}r_k=+\infty,\text{and}\quad H(r_k)<\frac{o(r_k)}{r_k}$$

Otherwise, there exists $\epsilon > 0$ and $\tilde{r} > 0$ such that $H(r) > \frac{\epsilon}{r}$ for $r > \tilde{r}$, but then $\int_{\mathbf{R}^2} |\nabla v^i|^2 \, dx = \infty$. Thus (24) implies

$$\int_{\partial B_{r_k}} |\nabla v^i| d\sigma \leq (2\pi r_k H(r_k))^{1/2} \leq (2\pi o(r_k))^{1/2}.$$

Therefore

$$\lim_{k\to\infty} \int_{\partial B_{r_k}} |\nabla v^i| d\sigma = 0.$$

Choose $R = r_k$, and let $k \to \infty$ in (23), then we have

$$\lim_{k\to\infty} \int_{B_{r_k}} \left[1 - e^{v^i+f} + \frac{\kappa}{q} A_0^i + \frac{d}{2q^2}(v^i - v^{i+1}) \right] \, dx = \frac{1}{2q^2} \int_{\mathbf{R}^2} g \, dx.$$

This, together with

$$\int_{\mathbf{R}^2} g dx = \sum_{j=1}^{m} 8\pi n_j \int_0^\infty \frac{r dr}{(1+r^2)^2} = 4\pi \sum_{j=1}^{m} n_j$$

completes the proof of the lemma. ∎

As a corollary of Lemma 4, we can get the following uniform bound.

Corollary 1 *Let (v^i, A_0^i) be as in Definition 1, and define*

$$S = \sup_{i\geq 1} \left(\int_{\mathbf{R}^2} e^{v^i+f} |\nabla(v^i+f)|^2 dx \right)^{\frac{1}{2}} \tag{25}$$

Then $S \leq (4\pi \sum_{j=1}^m n_j)^{\frac{1}{2}}$.

Proof: Multiplying (19) by e^{v^i+f} and integrating by parts, we have

$$\begin{aligned}
\int_{\mathbf{R}^2} e^{v^i+f} |\nabla(v^i+f)|^2 dx &= \int_{\mathbf{R}^2} \Big[d(v^{i-1} - v^i) e^{v^i+f} \\
&\quad + 2q^2 e^{v^i+f} (1 - e^{v^{i-1}+f} + \frac{\kappa}{q} A_0^{i-1}) \Big] dx \\
&\leq \int_{\mathbf{R}^2} \Big[d(v^{i-1} - v^i) + 2q^2 (1 - e^{v^{i-1}+f} + \frac{\kappa}{q} A_0^{i-1}) \Big] dx \\
&\leq 4\pi \sum_{j=1}^{m} n_j
\end{aligned}$$

by Lemma 4. $\square$

We now prove our main lemma in this section.

Proof of Lemma 3: From (16) we have

$$\begin{aligned}\mathcal{F}(v^{i-1})-\mathcal{F}(v^i) &= \int_{\mathbf{R}^2}\Big[\frac{1}{2}|\Delta(v^i-v^{i-1})|^2-\Delta v^i\Delta(v^i-v^{i-1})\\ &\quad +(\Delta g-\kappa^2 g)(v^i-v^{i-1})+\frac{\kappa^2}{2}|\nabla(v^i-v^{i-1})|^2\\ &\quad -\kappa^2\nabla(v^i-v^{i-1})\cdot\nabla v^i+II+III\Big]dx,\end{aligned}$$

where

$$\begin{aligned} II &= -2q^4\Big[(e^{v^i+f}-1)^2-(e^{v^{i-1}+f}-1)^2\Big]\\ III &= -2q^2\Big[e^{v^i+f}|\nabla(v^i+f)|^2-e^{v^{i-1}+f}|\nabla(v^{i-1}+f)|^2\Big].\end{aligned}$$

We also set

$$I=-\Delta v^i\Delta(v^i-v^{i-1})+(\Delta g-\kappa^2 g)(v^i-v^{i-1})-\kappa^2\nabla(v^i-v^{i-1})\cdot\nabla v^i.$$

Then

$$\begin{aligned}\mathcal{F}(v^{i-1})-\mathcal{F}(v^i) &= \int_{\mathbf{R}^2}\Big[\frac{1}{2}|\Delta(v^i-v^{i-1})|^2+\frac{\kappa^2}{2}|\nabla(v^i-v^{i-1})|^2\\ &\quad +I+II+III\Big]dx. \end{aligned} \tag{26}$$

We firstly estimate I. From (19) we have

$$A_0^{i-1}=\frac{1}{2q\kappa}\Big(2q^2(e^{v^{i-1}+f}-1)+g-d(v^{i-1}-v^i\Big)-\Delta v^i).$$

Putting this into (20) after substituting i with $i-1$ in (20), we have

$$\begin{aligned}&\Delta^2 v^i-(d+\kappa^2+2q^2e^{v^{i-1}+f})\Delta v^i+(d\kappa^2+2q^2de^{v^{i-1}+f})v^i\\ &\quad =(2q^2e^{v^{i-1}+f}-d)\Delta v^{i-1}+2q^2|\nabla(v^{i-1}+f)|^2e^{v^{i-1}+f}\\ &\qquad -4q^2ge^{v^{i-1}+f}+\Delta g-\kappa^2 g+d(\kappa^2+2q^2e^{v^{i-1}+f})v^{i-1}\\ &\qquad\quad -4q^4e^{v^{i-1}+f}(e^{v^{i-1}+f}-1)-2q\kappa d(A_0^{i-1}-A_0^{i-2}).\end{aligned}\tag{27}$$

Multiplying (27) by $v^i - v^{i-1}$, integrating by parts, we have

$$\int_{\mathbf{R}^2} \Big[\Delta v^i \Delta(v^i - v^{i-1}) + \kappa^2 \nabla(v^i - v^{i-1}) \cdot \nabla v^i - (\Delta g - \kappa^2 g)(v^i - v^{i-1})\Big]\, dx$$
$$= -\int_{\mathbf{R}^2} \Big[d|\nabla(v^i - v^{i-1})|^2 + (d\kappa^2 + 2q^2 d e^{v^{i-1}+f})(v^i - v^{i-1})^2$$
$$+2q\kappa d(A_0^{i-1} - A_0^{i-2})(v^i - v^{i-1}) - IV - V\Big]\, dx,$$

where we set

$$\begin{aligned} IV &= -4q^4 e^{v^{i-1}+f}(e^{v^{i-1}+f} - 1)(v^i - v^{i-1}) \\ V &= 2q^2\Big[e^{v^{i-1}+f}(v^i - v^{i-1})\Delta(v^i + v^{i-1}) \\ &\quad +|\nabla(v^{i-1} + f)|^2 e^{v^{i-1}+f}(v^i - v^{i-1}) + 2g e^{v^{i-1}+f}(v^i - v^{i-1})\Big]. \end{aligned}$$

Recalling the definition of I and observing $(A_0^{i-1} - A_0^{i-2})(v^i - v^{i-1}) \geq 0$, we get

$$I + IV + V \geq d|\nabla(v^i - v^{i-1})|^2 + (d\kappa^2 + 2q^2 d e^{v^{i-1}+f})(v^i - v^{i-1})^2. \quad (28)$$

To calculate $I + II + III$, we observe

$$II - IV = 4q^4(v^i - v^{i-1})\Big[e^{v^{i-1}+f}(e^{v^{i-1}+f} - 1) - e^{\lambda+f}(e^{\lambda+f} - 1)\Big]$$
$$= 4q^4(v^i - v^{i-1})(v^{i-1} - \lambda)e^{\eta+f}(2e^{\eta+f} - 1),$$

where we used mean value theorem repeatedly with $v^i \leq \eta \leq \lambda \leq v^{i-1}$. Thus

$$II - IV \geq -4q^4(v^i - v^{i-1})^2 e^{v^{i-1}+f}. \quad (29)$$

Now we have

$$\begin{aligned} III &= -2q^2[(e^{v^i+f} - e^{v^{i-1}+f})|\nabla(v^i + f)|^2 \\ &\quad +e^{v^{i-1}+f}(|\nabla(v^i + f)|^2 - |\nabla(v^{i-1} + f)|^2)] \\ &= -2q^2[e^{\lambda+f}(v^i - v^{i-1})|\nabla(v^i + f)|^2 \\ &\quad +e^{v^{i-1}+f}\nabla(v^i - v^{i-1}) \cdot \nabla(v^i + v^{i-1} + 2f)] \\ &= VI + VII, \\ V &= 2q^2 e^{v^{i-1}+f}(v^i - v^{i-1})[\Delta(v^i + v^{i-1} + 2f) \\ &\quad +|\nabla(v^{i-1} + f)|^2] \\ &= VIII + IX, \end{aligned}$$

where $v^{i-1} \geq \lambda \geq v^i$ by mean value theorem. We now calculate $III - V = (VII - VIII) - IX + VI$. By integration by parts we obtain

$$\begin{aligned}
&\int_{\mathbf{R}^2} [\ VII - VIII\ ] dx \\
&\quad = 2q^2 \int_{\mathbf{R}^2} \left[e^{v^{i-1}+f} (v^i - v^{i-1}) \nabla (v^{i-1} + f) \cdot \nabla (v^i + v^{i-1} + 2f) \right] dx \\
&\quad = \int_{\mathbf{R}^2} [X] dx, \\
&X - IX = 2q^2 e^{v^{i-1}+f} (v^i - v^{i-1}) \nabla (v^i + f) \cdot \nabla (v^{i-1} + f) \\
&\quad = XI.
\end{aligned}$$

Since $VI \leq 0$, we have

$$\begin{aligned}
VI + XI \quad \geq \quad & -2q^2 (v^i - v^{i-1}) \Big[e^{v^i+f} \nabla (v^i + f) \cdot \nabla (v^i - v^{i-1}) \\
& + (e^{v^i+f} - e^{v^{i-1}+f}) \nabla (v^i + f) \cdot \nabla (v^{i-1} + f) \Big] \\
\geq \quad & -2q^2 |v^i - v^{i-1}| \Big[e^{v^i+f} |\nabla (v^i + f)| |\nabla (v^i - v^{i-1})| \\
& + |e^{v^i+f} - e^{v^{i-1}+f}| |\nabla (v^{i-1} + f)|^2 \\
& + |e^{v^i+f} - e^{v^{i-1}+f}| |\nabla (v^i - v^{i-1})| |\nabla (v^{i-1} + f)| \Big] \\
\geq \quad & -2q^2 |v^i - v^{i-1}| \Big[(e^{v^i+f} |\nabla (v^i + f)| \\
& + e^{v^{i-1}+f} |\nabla (v^i - v^{i-1})| |\nabla (v^{i-1} + f)|) \\
& + e^{\lambda+f} |v^i - v^{i-1}| |\nabla (v^{i-1} + f)|^2 \Big]
\end{aligned}$$

by the mean value theorem where we used the fact $|e^{v^i+f} - e^{v^{i-1}+f}| \leq e^{v^{i-1}+f}$ in the last step. We use Hölder's inequality and interpolation inequality to obtain

$$\begin{aligned}
\int_{\mathbf{R}^2} [\ VI + XI\ ] dx \quad \geq \quad & -2q^2 \Big(\| e^{v^i+f} \nabla (v^i + f) \|_{L^2(\mathbf{R}^2)} \\
& + \|\ e^{v^{i-1}+f} \nabla (v^{i-1} + f) \|_{L^2(\mathbf{R}^2)} \Big) \\
& \times \| (v^i - v^{i-1}) \|_{L^\infty(\mathbf{R}^2)} \| \nabla (v^i - v^{i-1}) \|_{L^2(\mathbf{R}^2)} \\
& -2q^2 \| v^i - v^{i-1} \|^2_{L^\infty(\mathbf{R}^2)} \int_{\mathbf{R}^2} e^{v^i+f} |\nabla (v^i + f)|^2 dx
\end{aligned}$$

$$
\begin{aligned}
&\geq -4q^2CS\|v^i - v^{i-1}\|^{\frac{1}{2}}_{L^2(\mathbf{R}^2)}\|\Delta(v^i - v^{i-1})\|^{\frac{1}{2}}_{L^2(\mathbf{R}^2)} \\
&\quad \times\|\nabla(v^i - v^{i-1})\|_{L^2(\mathbf{R}^2)} \\
&\quad -2q^2CS^2\|v^i - v^{i-1}\|_{L^2(\mathbf{R}^2)}\|\Delta(v^i - v^{i-1})\|_{L^2(\mathbf{R}^2)},
\end{aligned}
$$

where C is an absolute constant and we set

$$
S = \sup_{i\geq 1}\left(\int_{\mathbf{R}^2} e^{v^i+f}|\nabla(v^i+f)|^2dx\right)^{\frac{1}{2}}
$$

as in Corollary 1. Applying Young's inequality, we have

$$
\begin{aligned}
\int_{\mathbf{R}^2}[\ III - V\ ]dx \;\geq\; & -Cq^4(S^2+S^4)\|v^i - v^{i-1}\|^2_{L^2(\mathbf{R}^2)} \\
& -Cq^2S\|\nabla(v^i - v^{i-1})\|^2_{L^2(\mathbf{R}^2)} \\
& -\frac{1}{4}\|\Delta(v^i - v^{i-1})\|^2_{L^2(\mathbf{R}^2)} \qquad (30)
\end{aligned}
$$

Combining with (28), (29) and (30), (26) becomes

$$
\begin{aligned}
\mathcal{F}(v^{i-1}) - \mathcal{F}(v^i) \;\geq\; & \frac{1}{4}\|\Delta(v^i - v^{i-1})\|^2_{L^2(\mathbf{R}^2)} + (d\kappa^2 - C)\|v^i - v^{i-1}\|^2_{L^2(\mathbf{R}^2)} \\
& +(d-C)\|\nabla(v^i - v^{i-1})\|^2_{L^2(\mathbf{R}^2)},
\end{aligned}
$$

where C is an absolute constant depending on q and $\sum_{j=1}^m n_j$. Taking d large enough, and using the Calderon-Zygmund inequality, we have finally

$$
\mathcal{F}(v^{i-1}) - \mathcal{F}(v^i) \geq C\|v^i - v^{i-1}\|_{H^2(\mathbf{R}^2)},
$$

which is a stronger form of (22). This completes the proof of Lemma 3. ∎

Corollary 2 *Let v be any admissible topological solution of (12)-(13), and v_a^q be the finite energy solution of the Abelian Higgs system. Then, we have*

$$
\mathcal{F}(v) \leq \mathcal{F}(v_a^q).
$$

Proof: Just substitute $v^i = v, v^{i-1} = v_a^q$ in the proof of Lemma 3, and instead of Lemma 4 we use

$$
\int_{\mathbf{R}^2}\left[|1 - e^{v+f} + \frac{\kappa}{q}A_0|\right]dx = \frac{1}{2q^2}\int_{\mathbf{R}^2} g = \frac{2\pi}{q^2}\sum_{j=1}^m n_j,
$$

which follows immediately from integration of (12) and Proposition 1.

□

5 Existence of Admissible Solutions and Asymptotic Decay

Based on the previous estimates for the iteration sequence $\{v^i\}$, in this section, we prove the existence of admissible topological solutions of our Bogomol'nyi equations (12)-(13). We also establish asymptotic exponential decay estimates of these solutions as $|z| \to \infty$. As a corollary of these decay estimates we prove that the action (2) and, hence the energy functional (3) are finite. Firstly we prove

Theorem 2 *Given $z_j \in \mathbf{R}^2$, $n_j \in \mathbf{Z}_+$ with $j = 1, \cdots, m$, there exists a smooth solution (ϕ, A, N) to (6)-(9) such that $\phi = 0$ at each $z = z_j$ with corresponding winding numbers n_j, and satisfying*

$$0 \le 1 - |\phi|^2 \le \frac{\kappa}{q} N = -\frac{\kappa}{q} A_0 \tag{31}$$

for all $q, \kappa > 0$

Proof: By (22) the monotone decreasing sequence $\{v^i\}$ satisfies

$$\mathcal{F}(v^i) \le \mathcal{F}(v^0) < \infty \quad \forall i = 1, 2, \cdots.$$

This implies by (17) and (16) that

$$\|v^i\|_{H^2(\mathbf{R}^2)} < C\mathcal{F}(v^i) \le C\mathcal{F}(v^0) \quad \forall i = 1, 2, \cdots. \tag{32}$$

Thus

$$\sup_{i \ge 0} \|v^i\|_{H^2(\mathbf{R}^2)} < \infty.$$

On the other hand, from (19)

$$A_0^i = \frac{1}{2q\kappa}\left[2q^2(e^{v^i+f} - 1) + g + d(v^{i+1} - v^i) - \Delta v^{i+1}\right]$$

which belongs to $L^2(\mathbf{R}^2)$ uniformly by (32).
Thus, $\sup_{i\ge0} \|A_0^i\|_{L^2(\mathbf{R}^2)} < \infty$, and $\sup_{i\ge0} \|\Delta A_0^i\|_{L^2(\mathbf{R}^2)} < \infty$ by (20).

Combining this with the Calderon-Zygmund inequality and the standard interpolation inequality, we obtain

$$\sup_{i\geq 0}\|A_0^i\|_{H^2(\mathbf{R}^2)}<\infty$$

Thus there exists $v, A_0 \in H^2(\mathbf{R}^2)$ and a subsequence (v^i, A_0^i) such that

$$v^i \to v \quad \text{and} \quad A_0^i \to A_0$$

both weakly in $H^2(\mathbf{R}^2)$ and strongly both in $H^1_{loc}(\mathbf{R}^2)$ and in $L^\infty_{loc}(\mathbf{R}^2)$ by Rellich's compactness theorem. The limits $v, A_0 \in H^2(\mathbf{R}^2)$ satisfies (12)-(13) in the weak sense, and by repeatedly using the standard linear elliptic regularity result we have $v, A_0 \in C^\infty(\mathbf{R}^2)$. Moreover, by construction we have

$$v \leq v_a^q \leq -f, \tag{33}$$

and by Proposition 1

$$\frac{q}{\kappa}(e^{v+f}-1) \leq A_0 \leq 0. \tag{34}$$

We define

$$N=-A_0, \quad \phi = \exp\frac{1}{2}(v+f+i\theta)$$

where $\theta = \sum_{j=1}^m 2n_j arg(z-z_j)$. For for $\alpha = \frac{1}{2}(A_1 - iA_2)$ and $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$ we also define

$$\alpha = i\partial_z ln\overline{\phi}.$$

Explicit computation shows

$$A_1 = \frac{1}{2}(\partial_2 v + \partial_2 b), \tag{35}$$

and

$$A_2 = \frac{1}{2}(-\partial_1 v - \partial_1 b), \tag{36}$$

where we set

$$b = -\sum_{j=1}^m n_j \ln(1+|z-z_j|^2).$$

Converting our reduction procedure from (12)-(13) to the Bogomol'nyi equations (6)-(9), we we find that the fields A_μ, ϕ, N $(\mu = 0,1,2)$

satisfy the Bogomol'nyi equations (6)-(9). In particular (32) follows immediately from (34) and (35). $\square$

We now establish asymptotic exponential decay estimates for admissible topological solutions of our Bogomol'nyi equations.

Theorem 3 *Let (ϕ, A_0, N) be any admissible topological solution of the Bogomol'nyi equations (6)-(9). Suppose $\epsilon > 0$ is given, then there exists $r_0 = r_0(\epsilon) > 0$ and $C = C_\epsilon$ such that*

$$\begin{aligned} 0 \leq 1 - |\phi|^2,\ |N|,\ |F_{12}| &\leq C_\epsilon e^{-q(1-\gamma-\epsilon)^{\frac{1}{2}}|z|} &(37)\\ |D_\mu \phi|,\ |\nabla A_0| &\leq C_\epsilon e^{-q(1-\gamma-\epsilon)^{\frac{1}{2}}|z|} &(38)\end{aligned}$$

if $|z| > r_0$. Here we set $\gamma = \frac{-\rho^2+\rho\sqrt{\rho^2+8}}{4}$ and $\rho = \frac{\kappa}{q}$.

Remark: We note that γ was chosen(see the proof below) so that $\frac{\rho^2}{\rho^2+2\gamma} = \gamma$, thus $0 < \gamma < 1$.

Proof of Theorem 3: From (10) and (11),

$$\begin{aligned} \Delta u^2 &= 2|\nabla u|^2 + 2u\Delta u \\ &\geq 4q^2(e^u - 1)u - 4q\kappa A_0 u, \\ \Delta A_0^2 &= 2|\nabla A_0|^2 + 2A_0 \Delta A_0 \\ &\geq 2(\kappa^2 + 2q^2 e^u)A_0^2 + 2\kappa q(1 - e^u)A_0 \end{aligned}$$

for $|z| > \sup_j\{|z_j|\}$. Let $E = u^2 + 2A_0^2$, then we have

$$\begin{aligned} \Delta E &\geq 4q^2\left(\frac{e^u - 1}{u}u^2 + 2e^u A_0^2\right) + 4\kappa^2 A_0^2 + 4\kappa q A_0(1 - e^u - u) \\ &\geq 4q^2\xi(u)E + 4\kappa^2 A_0^2 - 8\kappa q A_0 u &(39)\end{aligned}$$

where we used the inequality $t \leq e^t - 1$, and set $\xi(t) = \min\{e^t,\ \frac{e^t-1}{t}\}$. Note that

$$\begin{aligned} \xi(t) &= e^t \quad \text{if } t < 0 \\ &= \frac{e^t - 1}{t} \quad \text{otherwise} \end{aligned}$$

Also, $\xi > 0$ and $\xi(t) \to 1$ as $t \to 0$. The last term in (39) is estimated by

$$\begin{aligned} 8\kappa q|A_0 u| &\leq 4(\kappa^2 + 2\gamma q^2)A_0^2 + \frac{4\kappa^2}{\kappa^2 + 2\gamma q^2} q^2 u^2 \\ &= 4(\kappa^2 + 2\gamma q^2)A_0^2 + \frac{4\rho^2}{\rho^2 + 2\gamma} q^2 u^2 = 4\kappa^2 A_0^2 + 4\gamma q^2 E, \end{aligned}$$

since $\frac{\rho^2}{\rho^2+2\gamma} = \gamma$. Thus, (39) becomes

$$\Delta E \geq 4(\xi(u) - \gamma)q^2 E \tag{40}$$

Since $u \to 0$ as $|z| \to \infty$, given $\epsilon > 0$, we can choose r_0 so large that $\xi(u) \geq 1 - \epsilon$ on $|z| > r_0$. Thus, by comparing E with the function $\beta(z) = C_\epsilon e^{-2q(1-\gamma-\epsilon)^{\frac{1}{2}}|z|}$ in $|z| \geq r_0$, using the maximum principle, we deduce

$$|u|^2,\ |A_0|^2 \leq C_\epsilon e^{-2q(1-\gamma-\epsilon)^{\frac{1}{2}}|z|}$$

on $|z| > r_0$, where C_ϵ was fixed to compare E with β on $\{|z| = r_0\}$. (37) follows from the fact

$$0 \leq 1 - |\phi|^2 = 1 - e^u \leq |u|, \quad N = -A_0,$$

and

$$|F_{12}| \leq q(1 - |\phi|^2) + \kappa|N|,$$

which follows from (8). Next, we estimate the asymptotic decay of $|D_i\phi|^2$. We observe

$$\begin{aligned} |D_\mu\phi|^2 &= |(\partial_\mu - iqA_\mu)\phi|^2 \\ &= \frac{1}{4}e^u|(\partial_\mu(u + iq\theta) - iq(\partial_\mu^\perp u + \partial_\mu\theta)|^2 \\ &= \frac{1}{4}e^u|\partial_\mu u - iq\partial_\mu^\perp u|^2, \end{aligned}$$

where we used the notation $(\partial_\mu^\perp) = (\partial_0, -\partial_2, \partial_1)$. Thus $|D_\mu\phi|^2 \leq C|\nabla u|^2$. Therefore it is sufficient to have decay estimate for $|\nabla u|^2$. A direct calculation gives

$$\begin{aligned} \Delta|\nabla u|^2 &= 2|\nabla^2 u|^2 + 2\nabla u \cdot \nabla\Delta u \\ &\geq 4q^2\nabla u \cdot \nabla(e^u - 1 - \frac{\kappa}{q}A_0) \\ &= 4q^2 e^u|\nabla u|^2 - 4q\kappa\nabla u \cdot \nabla A_0 \\ \Delta|\nabla A_0|^2 &= 2|\nabla^2 A_0|^2 + 2\nabla A_0 \cdot \nabla\Delta A_0 \\ &\geq 2(\kappa^2 + 2q^2 e^u)|\nabla A_0|^2 - (2q\kappa e^u - 4q^2 e^u A_0)\nabla A_0 \cdot \nabla u \end{aligned}$$

for $|z| > \sup_j\{|z_j|\}$. We set $J = |\nabla u|^2 + 2|\nabla A_0|^2$, then we have

$$\begin{aligned}\Delta J &\geq 4q^2 e^u J + 4\kappa^2|\nabla A_0|^2 - (8\kappa q + 4q^2 e^u|A_0|)|\nabla u||\nabla A_0| \\ &\geq 4q^2 e^u(1 - \frac{1}{2}|A_0|)J + 4\kappa^2|\nabla A_0|^2 - 8\kappa q|\nabla u||\nabla A_0|, \quad (41)\end{aligned}$$

where we used $|\nabla u||\nabla A_0| \leq 1/2J$ in the second inequality. (41) is the same form as (39), observing in case of (41) we have

$$e^u(1 - \frac{1}{2}|A_0|) \to 1 \quad \text{as} \quad |z| \to \infty.$$

Given $\epsilon > 0$, we apply Young's inequality to the term $|\nabla u||\nabla A_0|$ similarly to the previous case, and get

$$\Delta J \geq 4q^2(1 - \gamma - \epsilon)J$$

when $|z| > r_0$ for r_0 large enough. The above equation is the same as (40). Thus J satisfies the estimate (38).

This completes the proof of Theorem 5. $\square$

Now we complete proof of our existence theorem by proving that the solutions constructed in Theorem 2 make our action in (2) finite. This follows if we prove that any admissible topological solution makes the action finite.

Corollary 3 *Let (A, ϕ, N) be the solution of the Bogomol'nyi equations (6)- (9) constructed Theorem 2, then we have*

$$\mathcal{A} = \mathcal{A}(A, \phi, N) < \infty.$$

Proof: Since $N = -A_0 \in H^1(\mathbf{R}^2)$ and $|\phi| \leq 1$,

$$\int_{\mathbf{R}^2} (\partial_\mu N)^2 dx < \infty, \quad \int_{\mathbf{R}^2} N^2|\phi|^2 dx < \infty.$$

From (4) and (10) we have $F_{12} = q|\phi|^2 - \kappa N - q \in L^2(\mathbf{R}^2)$. Clearly

$$F_{0i} = -\partial_i A_0 \in L^2(\mathbf{R}^2), \quad i = 1,\ 2.$$

Therefore

$$\int_{\mathbf{R}^2} |F_{\mu\nu}|^2 dx < \infty.$$

We now consider the Chern-Simons term. Firstly we have

$$\int_{\mathbf{R}^2} |F_{12}A_0|dx \leq \left(\int_{\mathbf{R}^2} |F_{12}|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^2} |A_0|^2 dx\right)^{\frac{1}{2}} < \infty.$$

Thus it suffices to prove

$$F_{01}A_2 = -\partial_1 A_0 A_2, \quad F_{02}A_1 = \partial_2 A_0 A_1 \in L^1(\mathbf{R}^2). \tag{42}$$

Since $A_1, A_2 \in L^p(\mathbf{R}^2)$ for all $p \in (2, \infty]$ from (35)- (36), and $\nabla A_0 \in L^q(\mathbf{R}^2)$, for $q \in [1, \infty]$ from (38), (42) follows immediately by the Hölder inequality. Therefore we also have that

$$\epsilon^{\mu\nu\lambda} F_{\mu\nu} A_\lambda \in L^1(\mathbf{R}^2).$$

Finally from (38) we have

$$|D_\mu \phi|^2 \in L^1(\mathbf{R}^2).$$

This completes the proof of the corollary. $\square$

6 Abelian Higgs Limit

In this section we prove that, for q fixed, the sequence of admissible topological solutions, $(v^{\kappa,q}, A_0^{\kappa,q})$ converges to $(v_a^q, 0)$, as κ goes to zero, where v_a^q is the finite energy solution of the Abelian Higgs system. Firstly we establish:

Lemma 5 *Let $(v^{\kappa,q}, A_0^{\kappa,q})$ be any admissible topological solution of (12) and (13). Then, for each fixed $q \in (0, \infty)$, we have*

$$\sup_{0<\kappa<1} \|v^{\kappa,q}\|_{H^2(\mathbf{R}^2)} < \infty, \sup_{0<\kappa<1} \|A_0^{\kappa,q}\|_{H^2(\mathbf{R}^2)}. < \infty \tag{43}$$

Thus, by the Sobolev embedding we have

$$\sup_{0<\kappa<1} \|v^{\kappa,q}\|_{L^\infty(\mathbf{R}^2)} < \infty, \sup_{0<\kappa<1} \|A_0^{\kappa,q}\|_{L^\infty(\mathbf{R}^2)} < \infty \tag{44}$$

Proof: Let $\kappa \in (0,1)$. From Corollary 2 and (17) we have

$$\begin{aligned}\|v^{\kappa,q}\|_{H^2(\mathbf{R}^2)} &\leq (1+\kappa^2)C_1 + C_2\mathcal{F}(v^{\kappa,q}) \\ &\leq (1+\kappa^2)C_1 + C_2\mathcal{F}(v_a^q) \leq C(1+\kappa^2),\end{aligned}$$

where C_1, C_2 and C are constants independent of κ. Thus the first inequality of (43) follows. Now, taking $L^2(\mathbf{R}^2)$ inner product (13) with $A_0^{\kappa,q}$, we have after integration by part

$$\begin{aligned}\int_{\mathbf{R}^2} |\nabla A_0^{\kappa,q}|^2 + (\kappa^2 + 2q^2 e^{v^{\kappa,q}+f})|A_0^{\kappa,q}|^2 \, dx &= \int_{\mathbf{R}^2} \kappa q(1 - e^{v^{\kappa,q}+f})|A_0^{\kappa,q}| \, dx \\ &\leq \frac{\kappa^2}{2}\int_{\mathbf{R}^2} |A_0^{\kappa,q}|^2 dx + \frac{q^2}{2}\int_{\mathbf{R}^2} (1 - e^{v^{\kappa,q}+f})^2 \, dx \\ &\leq \frac{\kappa^2}{2}\int_{\mathbf{R}^2} |A_0^{\kappa,q}|^2 dx + \frac{q^2}{2}\int_{\mathbf{R}^2} |v^{\kappa,q} + f|^2 \, dx.\end{aligned}$$

Thus, by Young's inequality and the first inequality in (43) we obtain

$$\int_{\mathbf{R}^2} |\nabla A_0^{\kappa,q}|^2 dx + \int_{\mathbf{R}^2} (\kappa^2 + 4q^2 e^{v^{\kappa,q}+f})|A_0^{\kappa,q}|^2 dx \leq C, \tag{45}$$

where C is independent of κ. From (13) and (45) we have

$$\begin{aligned}\int_{\mathbf{R}^2} |\Delta A_0^{\kappa,q}|^2 \, dx &\leq 2\kappa^2 q^2 \int_{\mathbf{R}^2} (1 - e^{v^{\kappa,q}+f})^2 \, dx + 2\int_{\mathbf{R}^2} (\kappa^2 + 2q^2 e^{v^{\kappa,q}+f})^2 |A_0^{\kappa,q}|^2 \, dx \\ &\leq 2\kappa^2 q^2 \int_{\mathbf{R}^2} |v^{\kappa,q} + f|^2 e^{2(\lambda+f)} \, dx + 2(\kappa^2 + 2q^2)\int_{\mathbf{R}^2} (\kappa^2 + 4q^2 e^{v^{\kappa,q}+f})|A_0^{\kappa,q}|^2 \, dx \\ &\leq 2q^2 \int_{\mathbf{R}^2} |v^{\kappa,q} + f|^2 dx + 2(1+q^2)\int_{\mathbf{R}^2} (\kappa^2 + 4q^2 e^{v^{\kappa,q}+f})|A_0^{\kappa,q}|^2 \, dx \leq C\end{aligned}$$

for a constant C independent of κ, where $\lambda \in (v^{\kappa,q} + f, 0)$, and we used the mean value theorem. Thus, by the Calderon-Zygmund inequality

$$\int_{\mathbf{R}^2} |D^2 A_0^{\kappa,q}|^2 dx \leq C. \tag{46}$$

By Sobolev's embedding for a bounded domain, for any ball $B_R = \{|z| < R\} \subset \mathbf{R}^2$, we have.

$$\|A_0^{\kappa,q}\|_{L^\infty(B_R)} \leq C,$$

where C is independent of κ. We take $R > \max_{1\leq j\leq m}\{|z_j| + 1\}$. Then,

$$\int_{\mathbf{R}^2} |A_0^{\kappa,q}|^2 dx = \int_{B_R} |A_0^{\kappa,q}|^2 dx + \int_{\mathbf{R}^2 - B_R} |A_0^{\kappa,q}|^2 dx$$

$$\leq \pi R^2 \|A_0^{\kappa,q}\|^2_{L^\infty(B_R)} + \|e^{v^{\kappa,q}+f}\|_{L^\infty(\mathbf{R}^2-B_R)} \int_{\mathbf{R}^2-B_R} e^{v^{\kappa,q}+f}|A_0^{\kappa,q}|^2 dx$$
$$\leq C_1(R) + C_2\|e^f\|_{L^\infty(\mathbf{R}^2-B_R)} \int_{\mathbf{R}^2} e^{v^{\kappa,q}+f}|A_0^{\kappa,q}|^2 dx \leq C(R), \quad (47)$$

where $C_1(R), C_2$ and $C(R)$ are independent of κ. Combining (45)-(47), we obtain the second inequality in (43). This completes the proof of Lemma 5. □

Theorem 4 *Let $v^{\kappa,q}, A_0^{\kappa,q}$ be the admissible topological solutions of (12) and (13), and v_a^q a finite energy solution of the Abelian Higgs system. Let q be fixed. For all $k \in Z_+$ we have*

$$v^{\kappa,q} \to v_a^q, \quad \textit{and} \quad A_0^{\kappa,q} \to 0 \quad \textit{in} \quad H^k(\mathbf{R}^2).$$

as $\kappa \to 0$.

Proof: We have by mean value theorem

$$\Delta(v^{\kappa,q} - v_a^q) = 2q^2(e^{v^{\kappa,q}+f} - e^{v_a^q+f}) - 2\kappa q A_0^{\kappa,q}$$
$$= 2q^2 e^{\lambda+f}(v^{\kappa,q} - v_a^q) - 2\kappa q A_0^{\kappa,q} \quad (48)$$

where $\lambda \in (v^{\kappa,q}, v_a^q)$. Multiplying (48) by $v^{\kappa,q} - v_a^q$, we have after integration by parts

$$\int_{\mathbf{R}^2} |\nabla(v^{\kappa,q} - v_a^q)|^2 + 2q^2 e^{\lambda+f}(v^{\kappa,q} - v_a^q)^2 \, dx = 2\kappa q \int_{\mathbf{R}^2} A_0^{\kappa,q}(v^{\kappa,q} - v_a^q) \, dx$$
$$\leq 2\kappa q \|A_0^{\kappa,q}\|_{L^2(\mathbf{R}^2)} \|v^{\kappa,q} - v_a^q\|_{L^2(\mathbf{R}^2)} \leq \kappa C$$

where C is independent of κ by Lemma 5. Since

$$\|\lambda\|_{L^\infty(\mathbf{R}^2)} \leq \|v^{\kappa,q}\|_{L^\infty(\mathbf{R}^2)} + \|v_a^q\|_{L^\infty(\mathbf{R}^2)} \leq C$$

independently of $\kappa < 1$, we have from the above estimate

$$\int_{\mathbf{R}^2} |\nabla(v^{\kappa,q} - v_a^q)|^2 + e^f |v^{\kappa,q} - v_a^q|^2 \, dx \to 0$$

as $\kappa \to 0$. Let $\Omega_\delta = \cup_{j=1}^m \{|z - z_j| < \delta\}$. Now,

$$\int_{\Omega_\delta} |v^{\kappa,q} - v_a^q|^2 dx \leq \pi m \delta^2 \|v^{\kappa,q} - v_a^q\|^2_{L^\infty(\mathbf{R}^2)} \leq C\delta^2.$$

where C is independent of κ by Lemma 5. Thus, for any given $\epsilon > 0$, we can choose δ independently of κ so that

$$\int_{\Omega_\delta} |v^{\kappa,q} - v_a^q|^2 dx \le \frac{\epsilon}{2}.$$

For such δ we have

$$\begin{aligned}\int_{\mathbf{R}^2} |v^{\kappa,q} - v_a^q|^2 dx &= \int_{\Omega_\delta} |v^{\kappa,q} - v_a^q|^2 dx + \int_{\mathbf{R}^2 - \Omega_\delta} |v^{\kappa,q} - v_a^q|^2 dx \\ &\le \frac{\epsilon}{2} + \sup_{\mathbf{R}^2 - \Omega_\delta} \{e^{|f|}\} \int_{\mathbf{R}^2} e^f |v^{\kappa,q} - v_a^q|^2 dx \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

for sufficiently small κ, i.e.

$$\int_{\mathbf{R}^2} |v^{\kappa,q} - v_a^q|^2 dx \to 0$$

as $\kappa \to 0$. Combining the above results, we obtain

$$v^{\kappa,q} \to v_a^q \ \text{ in } \ H^1(\mathbf{R}^2) \ \text{ as } \quad \kappa \to 0.$$

Now we prove the convergence for $A_0^{\kappa,q}$. Multiplying (13) by $A_0^{\kappa,q}$ and integrating, we estimate

$$\int_{\mathbf{R}^2} |\nabla A_0^{\kappa,q}|^2 + (\kappa^2 + 2q^2 e^{v^{\kappa,q}+f}) |A_0^{\kappa,q}|^2 \, dx \le \kappa q \int_{\mathbf{R}^2} (1 - e^{v^{\kappa,q}+f}) |A_0^{\kappa,q}| \, dx$$

$$\le \kappa q \|A_0^{\kappa,q}\|_{L^2(\mathbf{R}^2)} \|1 - e^{v^{\kappa,q}+f}\|_{L^2(\mathbf{R}^2)} \le C\kappa \|v^{\kappa,q} + f\|_{L^2(\mathbf{R}^2)} \le C\kappa.$$

where we used Lemma 5 in the first and third step and use the fact $1 - e^t \le t$ for $t \le 0$ in the second step. Using the fact $|v^{\kappa,q}| < C$ uniformly in $\kappa < 1$, we obtain from this

$$\int_{\mathbf{R}^2} |\nabla A_0^{\kappa,q}|^2 + e^f |A_0^{\kappa,q}|^2 \, dx \to 0$$

as $\kappa \to 0$. Since $|A_0^{\kappa,q}| < C$ uniformly in $\kappa < 1$, by Lemma 5 we can deduce $A_0^{\kappa,q} \to 0$ in $H^1(\mathbf{R}^2)$ similarly to the case of $v^{\kappa,q}$. From these results together with uniform bounds $\|v^{\kappa,q}\|, \|A_0^{\kappa,q}\| \le C$, applying the standard elliptic regularity to (48) and (13) repeatedly, we obtain

$$(v^{\kappa,q}, \ A_0^{\kappa,q}) \to (v_a^q, \ 0) \quad \text{in} \quad [H^k(\mathbf{R}^2)]^2, \quad \forall k \ge 1$$

$\Box$

7 Chern-Simons Limit

In this section we study the behaviors of $v^{\kappa,q}, A_0^{\kappa,q}$ as $\kappa, q \to \infty$ with $l = q^2/\kappa$ kept fixed for the admissible topological solutions. Although we could not obtain the strong convergence to a solution of the Chern-Simons equation, instead, we will prove that the sequence $\{v^{\kappa,q}\}$ is "weakly approximating" the Chern-Simons equation:

$$\Delta v = 4l^2 e^{v+f}(e^{v+f} - 1) + g.$$

We denote $l = q^2/\kappa$ the fixed number, and $\alpha^{\kappa,q} = qA_0^{\kappa,q}$ throughout this section.

Theorem 5 *Let $\{(v^{\kappa,q}, A_0^{\kappa,q})\}$ be a sequence of admissible topological solutions of (12)-(13). For any $\psi \in C_0^\infty(\mathbf{R}^2)$ we have*

$$\lim_{\kappa,q\to\infty} \int_{\mathbf{R}^2} \left[\Delta v^{\kappa,q} - 4l^2 e^{v^{\kappa,q}+f}(e^{v^{\kappa,q}+f} - 1) - g\right] \psi dx = 0. \qquad (49)$$

For proof of this theorem we firstly establish the following lemma which is interesting in itself.

Lemma 6 *Let $\{(v^{\kappa,q}, A_0^{\kappa,q})\}$ be given as in Theorem 5. For any fixed $p \in [1, \infty)$ we have*

$$\lim_{\kappa,q\to\infty} \|\alpha^{\kappa,q} - l(e^{v^{\kappa,q}+f} - 1)\|_{L^p(\mathbf{R}^2)} = 0.$$

Proof: Firstly we have from $v^{\kappa,q} + f \le 0$

$$\|e^{v^{\kappa,q}+f} - 1\|_{L^\infty(\mathbf{R}^2)} \le 1.$$

Also, from $0 \ge \alpha^{\kappa,q} \ge l(e^{v^{\kappa,q}+f} - 1)$,

$$\|\alpha^{\kappa,q}\|_{L^\infty(\mathbf{R}^2)} \le l\|e^{v^{\kappa,q}+f} - 1\|_{L^\infty(\mathbf{R}^2)} \le l.$$

From (12) we have

$$\begin{aligned} \int_{\mathbf{R}^2} |\alpha^{\kappa,q} - l(e^{v^{\kappa,q}+f} - 1)| dx &= \int_{\mathbf{R}^2} \left[\alpha^{\kappa,q} - l(e^{v^{\kappa,q}+f} - 1)\right] dx \\ &= \frac{1}{q}\int_{\mathbf{R}^2} g dx. \end{aligned}$$

Thus

$$\lim_{\kappa,q\to\infty} \|\alpha^{\kappa,q} - l(e^{v^{\kappa,q}+f} - 1)\|_{L^1(\mathbf{R}^2)} = 0.$$

By a standard interpolation inequality

$$
\begin{aligned}
&\|\alpha^{\kappa,q} - l(e^{v^{\kappa,q}+f} - 1)\|_{L^p(\mathbf{R}^2)} \\
&\quad\leq \quad \|\alpha^{\kappa,q} - l(e^{v^{\kappa,q}+f} - 1)\|^{\frac{1}{p}}_{L^1(\mathbf{R}^2)} \|\alpha^{\kappa,q} - l(e^{v^{\kappa,q}+f} - 1)\|^{1-\frac{1}{p}}_{L^\infty(\mathbf{R}^2)} \\
&\quad\leq \quad (2l)^{1-\frac{1}{p}} \|\alpha^{\kappa,q} - l(e^{v^{\kappa,q}+f} - 1)\|^{\frac{1}{p}}_{L^1(\mathbf{R}^2)} \to 0
\end{aligned}
$$

as $\kappa, q \to \infty$ with $q^2/\kappa = l$ fixed. □

Proof of Theorem 5: From (13) added by (14)$\times q/\kappa$ we obtain

$$
\Delta(v^{\kappa,q} + \frac{2l}{q}\alpha^{\kappa,q}) = g + 4le^{v^{\kappa,q}+f}\alpha^{\kappa,q}.
$$

Multiplying $\psi \in C_0^\infty(\mathbf{R}^2)$, and integrating by parts, we obtain

$$
\int_{\mathbf{R}^2} \Delta v^{\kappa,q}\psi = 4l^2 \int_{\mathbf{R}^2} \left[e^{v^{\kappa,q}+f}(e^{v^{\kappa,q}+f} - 1) + g \right] \psi dx
$$

$$
+\frac{2l}{q}\int_{\mathbf{R}^2} \alpha^{\kappa,q}\Delta\psi dx + 4l\int_{\mathbf{R}^2} e^{v^{\kappa,q}+f}[\alpha^{\kappa,q} - l(e^{v^{\kappa,q}+f} - 1)]\psi dx.
$$

Now we have

$$
\lim_{\kappa,q\to\infty} \left| \frac{2l}{q} \int_{\mathbf{R}^2} \alpha^{\kappa,q}\Delta\psi dx \right| \leq \lim_{\kappa,q\to\infty} \frac{2l^2}{q} \int_{\mathbf{R}^2} |\Delta\psi| dx = 0.
$$

and by Lemma 6

$$
\lim_{\kappa,q\to\infty} \int_{\mathbf{R}^2} e^{v^{\kappa,q}+f}[\alpha^{\kappa,q} - l(e^{v^{\kappa,q}+f} - 1)]dx
$$

$$
\leq \|\psi\|_{L^\infty(\mathbf{R}^2)} \lim_{\kappa,q\to\infty} \int_{\mathbf{R}^2} |\alpha^{\kappa,q} - l(e^{v^{\kappa,q}+f} - 1)| dx = 0.
$$

Thus, Theorem 5 follows. □

Remark: If we could have uniform $L^1(\mathbf{R}^2)$ estimate of $\nabla v^{\kappa,q}$, then we could prove existence of subsequence $\{v^{\kappa,q}\}$ and its $L^q_{loc}(\mathbf{R}^2)$ $(1 \leq q < 2)$-limit v such that v is a smooth solution of the Chern-Simons equation.

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